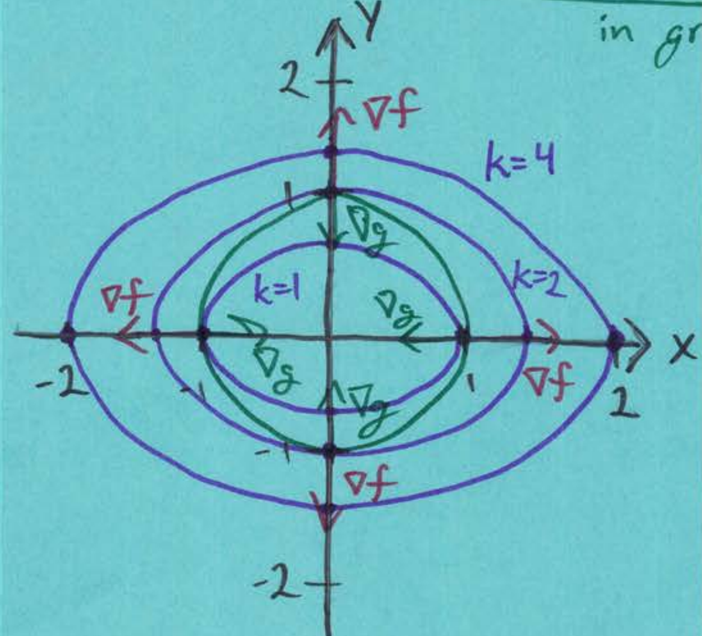
Lecture 21

14.8 - Lagrange Multipliers

Suppose we wanted to maximize the function $f(x,y) = x^2 + 2y^2$ subject to the constraint $x^2 + y^2 = 1$.

Let's approach this graphically by plotting level curves (in purple) of f, together with the constraint (in green):



The maxima of f on the constraint are the points where the constraint intersects the level curve for the largest k $(f(x,y)=k)$. We can see, then, that the maximum value is 2, similarly the minimum value is 1.

If we think of the constraint itself as a level curve, something interesting happens. Say $g(x,y) = -x^2 - y^2$, then the constraint is the level curve $g(x,y) = -1$.

Behold, we see that $\nabla f \parallel \nabla g$ at these extreme points! This is no coincidence. We formalize this now. We will do the work for a function of 3 variables, but the 2 variable case is similar.

Say we want to extremize $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$, a level surface of g. Suppose $P=(x_0,y_0,z_0)$ is an extreme point. Let $\vec{r}(t)=\langle x(t),y(t),z(t)\rangle$ be a curve in the level surface $g=k$, with $\vec{r}(t_0)=P$. Then we know $\nabla g(P) \perp \vec{r}\,'(t_0)$. Now, since we are extremizing f subject to $g=k$, we are essentially looking for the "higest and lowest" points on the graph of $f(x,y,z)$ over the surface $g(x,y,z)=k$. This means we can plug $\vec{r}(t)$ into f: $h(t)=f(x(t),y(t),z(t))$. Since $\vec{r}(t_0)=P$, $h(t)$ hits an extreme value at t_0, hence, from calc I we know $h'(t_0)=0$. By the chain rule

$$h'(t_0) = f_x(x_0,y_0,z_0)x'(t_0) + f_y(x_0,y_0,z_0)y'(t_0) + f_z(x_0,y_0,z_0)z'(t_0)$$
$$= \nabla f(x_0,y_0,z_0)\cdot \vec{r}\,'(t_0) = 0.$$

This is true for any curve in $g(x,y,z)=k$, so we have $\nabla f(x_0,y_0,z_0)$ is perpendicular to $g=k$ at P. But ∇g is also $\perp$ to $g=k$ at P, so we must have $\nabla f(x_0,y_0,z_0)=\lambda \nabla g(x_0,y_0,z_0)$.

λ is called a <u>Lagrange multiplier</u>.

We have:

<u>Method of Lagrange Multipliers</u>:

To find the extreme values of $f(x,y,z)$, subject to the constraint $g(x,y,z)=k$ (assuming the extreme values exist, and $\nabla g \neq \vec{0}$ on the surface $g(x,y,z)=k$):

ⓐ Find all values of x,y,z, and λ solving the system:

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z)=k \end{cases} \iff \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g(x,y,z)=k. \end{cases}$$

ⓑ Evaluate f at all points (x,y,z) found in ⓐ and identify the maxima/minima.

A word of caution!

This method requires solving an often nonlinear system of 3 (if f is a function of 2 variables) or 4 (if f is a function of 3 variables) equations. There is no "method" for solving them in general! This can sometimes require some ingenuity to solve!

Let's see an example in two variables first:

Ex: Find the extreme values of $f(x,y) = x^2 + 2y^2$ subject to the constraint $x^2 + y^2 = 1$.

Sol: Let $g(x,y) = x^2 + y^2$, then the constraint is $g = 1$.

$\nabla f = \langle 2x, 4y \rangle$, $\nabla g = \langle 2x, 2y \rangle$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases} \Rightarrow \begin{cases} 2x = \lambda 2x & (1) \\ 4y = \lambda 2y & (2) \\ x^2 + y^2 = 1 & (3) \end{cases}$$

(1) $\Rightarrow x = 0$ or $\lambda = 1$.

$\underline{x=0}$: (3) $\Rightarrow y = \pm 1 \overset{(2)}{\Rightarrow} \lambda = \pm 2$ (we don't actually need λ)

So, potential points are $(0,-1), (0,1)$.

$\underline{\lambda=1}$: (2) $\Rightarrow 4y = 2y \Rightarrow y = 0 \overset{(3)}{\Rightarrow} x = \pm 1$. Also check $(1,0), (-1,0)$

Candidates	Value of f
$(1,0)$	1
$(-1,0)$	1
$(0,1)$	2
$(0,-1)$	2

$\big\}$ min (for $(1,0)$, $(-1,0)$)

$\big\}$ max (for $(0,1)$, $(0,-1)$)

◇

Let's revisit the problem at the end of Monday:

<u>Ex</u>: A cardboard box without a lid is to have a volume of $32000\,cm^3$. Find the dimensions that minimize the amount of cardboard used.

<u>Sol</u>: We want to minimize $A = xy + 2(xz+yz)$ subject to the constraint $V = xyz = 32000$. Set up Lagrange multipliers:

$$\begin{cases} \nabla A = \lambda \nabla V \\ V = 32000 \end{cases} \Rightarrow \begin{cases} y + 2z = \lambda yz & ① \\ x + 2z = \lambda xz & ② \\ 2x + 2y = \lambda xy & ③ \\ xyz = 32000 & ④ \end{cases}$$

Since $x, y, z \neq 0$ (else we would have zero volume!), we can solve for λ with ①, ②, ③

$$\lambda \overset{①}{=} \underline{\frac{1}{z} + \frac{2}{y}} \overset{②}{=} \overline{\frac{1}{z} + \frac{2}{x}} \overset{③}{=} \underline{\overline{\frac{2}{y} + \frac{2}{x}}}$$

Remember, we don't actually need λ, it just has to be able to be solved for. The green equality gives: $\frac{1}{z}+\cancel{\frac{2}{y}}=\cancel{\frac{2}{y}}+\frac{2}{x} \Rightarrow \frac{1}{z}=\frac{2}{x} \Rightarrow x=2z$.

The red equality gives:

$$\frac{1}{z}+\cancel{\frac{2}{x}}=\frac{2}{y}+\cancel{\frac{2}{x}} \Rightarrow \frac{1}{z}=\frac{2}{y} \Rightarrow y=2z$$

So, $x=y=2z$. Plug this into 4:

$$xyz=(2z)(2z)z=4z^3=32000 \Rightarrow z^3=8000 \Rightarrow z=20$$

$\Rightarrow x=y=2(20)=40$. So $(40,40,20)$ is a potential minimum. We only obtained one solution, but we don't know whether it is a max or min (it is one of these two though)... to check, we just plug in some other point on $xyz=32000$ to A. If A at that point is larger, than at $(40,40,20)$, then $(40,40,20)$ is a min. One can similarly check for a max (the value should be less at other points). First, $A(40,40,20)=(40)(40)+2((40)(20)+(40)(20))$
$=1600+2(800+800)=4800$.

Another point on $xyz=32000$ is $(32,10,100)$, and
$A(32,10,100)=(32)(10)+2((32)(100)+(10)(100))=320+2(3200+1000)=8720$

Since $A(32,10,100)>A(40,40,20)$, $(40,40,20)$ is indeed a min. $\diamond$

Now, let's do a hard problem!

Ex: Find the extrema of $f(x,y)=e^{-xy}$ on the region $x^2+4y^2 \leq 1$.

Sol: First, we search for critical points inside:

$\nabla f = \langle -ye^{-xy}, -xe^{-xy} \rangle$. Since $e^{-xy}>0$ for all x & y, $\nabla f = \vec{0}$ if and only if $x=y=0$. So the only critical point inside is $(0,0)$.

Now, we check the boundary $x^2+4y^2=1$. We use Lagrange multipliers for this: Let $g(x,y)=x^2+4y^2$. Then we have:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g=1 \end{cases} \Rightarrow \begin{cases} -ye^{-xy} = \lambda 2x & (1) \\ -xe^{-xy} = \lambda 8y & (2) \\ x^2+4y^2 = 1 & (3) \end{cases}$$

First, let's try to solve for $-e^{-xy}$. If we use (1), then we need to make sure $y \neq 0$. If $y=0$, then (1) implies $\lambda=0$ or $x=0$. If $x=0$, this contradicts (3). If $\lambda=0$, then, by (2) $-xe^{-xy}=\lambda 8y=0 \Rightarrow x=0$ since $e^{-xy}\neq 0$. This again contradicts (3). We can play the same game to show $x\neq 0$. So, by (1), $-e^{-xy}=\frac{2\lambda x}{y}$. Plug this into (2):

$$x(-e^{-xy}) = x\left(\frac{2\lambda x}{y}\right) = \frac{2\lambda x^2}{y} = 8\lambda y \Rightarrow \lambda x^2 = 4\lambda y^2.$$

Now, we use ③ to solve for y:

③ $\Rightarrow x^2 = 1-4y^2$. So, the previous equation gives:

$$\lambda(1-4y^2) = \lambda(4y^2) \iff 8\lambda y^2 = \lambda.$$

Now, if $\lambda = 0$, then ① $\Rightarrow y = 0$, which we know cannot happen, so $\lambda \neq 0$. Thus $8y^2 = 1 \Rightarrow y = \pm\sqrt{\frac{1}{8}} = \pm\frac{1}{2\sqrt{2}}$.

③ $\Rightarrow x^2 = 1-4y^2 = 1-4\left(\frac{1}{8}\right) = 1-\frac{1}{2} = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

So, we have four points to check:

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right).$$

Candidates	Value of f
$(0, 0)$	1
$\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$	$e^{1/4}$ max
$\left(\frac{1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right)$	$e^{-1/4}$ min
$\left(\frac{-1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$	$e^{1/4}$ max
$\left(\frac{-1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right)$	$e^{-1/4}$ min

$e^{1/4} > 1$ since $e > 1$

$\Rightarrow e^{-1/4} = \frac{1}{e^{1/4}} < 1$.

$\diamond$